Chapter 9

Reconciling Inequalities

In This Chapter

- Going back and forth between inequality and interval notation
- Performing operations on inequalities and reversing the sense
- Dealing with fractional and multi-factored inequalities

Equality is an important tool in mathematics and science. This chapter introduces you to algebraic *inequality,* which isn't exactly the opposite of equality. You could say that algebraic inequality is a bit like equality but softer. You use inequality for comparisons — when you're determining if something is positive or negative, bigger than or smaller than, between numbers, or infinite. Inequality allows you to sandwich expressions between values on the low end and the high end.

Algebraic inequalities show relationships between a number and an expression or between two expressions. The inequality relation is a bit less than precise. One thing can be bigger by a lot or bigger by a little, but there's still that relationship between them — that one is bigger than the other.

Many operations involving inequalities work the same as operations on equalities and equations, but you need to pay attention to some important differences that I show you in this chapter.

Introducing Interval Notation

Algebraic operations and manipulations are performed on inequality statements while they're in an inequality format.

You see the inequality statements written using the following notation:

- $<$: Less than
- $>$: Greater than
- $\leq$: Less than or equal to
- $\geq$: Greater than or equal to

To keep the direction straight as to which way to point the arrow, just remember that the itsy-bitsy part of the arrow is next to the smaller (itsy-bitsier) of the two values.

Inequality statements have been around for a long time. The symbols are traditional and accepted by mathematicians around the world. But a new notation is gaining popularity, called *interval notation.* Interval notation uses parentheses and brackets instead of inequality symbols, and it introduces the infinity symbol.

Comparing inequality and interval notation

Before defining how interval notation is used, let me first write the same statement in both inequality and interval notation:

Inequality	***Interval Notation***
$x > 8$	$(8, \infty)$
$x < 2$	$(-\infty, 2)$
$x \geq -7$	$[-7, \infty)$
$x \leq 5$	$(-\infty, 5]$
$-4 < x \leq 10$	$(-4, 10]$

So, now that you've seen interval notation in action, let me give you the rules for using it.

ALGEBRA RULES

Interval notation expresses inequality statements with the following rules:

- Parentheses to show *less than* or *greater than* (but not including)

- Brackets to show *less than or equal to* or *greater than or equal to*
- Parentheses at both infinity or negative infinity
- Numbers and symbols written in the same left-to-right order as on a number line

Here are some examples of writing inequality statements using interval notation or vice versa:

- **$-3 \leq x \leq 11$ becomes $[-3, 11]$.**
- **$-4 \leq x < -3$ becomes $[-4, -3)$.**
- **$x > -9$ becomes $(-9, \infty)$.**
- **$5 < x$ becomes $(5, \infty)$.** Notice that the variable didn't come first in the inequality statement, and saying 5 must be smaller than some numbers is the same as saying that those numbers are bigger (greater) than 5, or $x > 5$.
- **$4 < x < 15$ becomes $(4, 15)$.** Here's my biggest problem with interval notation: The notation (4, 15) looks like a point on the coordinate plane, not an interval containing numbers between 4 and 15. You just have to be aware of the context when you come across this notation.
- **$[-8, 5]$ becomes $-8 \leq x \leq 5$.**
- **$(-\infty, 0]$ becomes $x \leq 0$.**
- **$(44, \infty)$ becomes $x > 44$.**

Graphing inequalities

One of the best ways of describing inequalities is with a graph. Graphs in the form of number lines are a great help when solving quadratic inequalities (see the "Taking on Quadratic and Rational Inequalities" section, later in this chapter).

A number-line graph of an inequality consists of numbers representing the starting and ending points of any interval described by the inequality and symbols above the numbers indicating whether the number is to be included in the answer. The symbols used with inequality notation are hollow circles and filled-in circles. The symbols used with interval notation are the same parentheses and brackets used in the statements.

Write the statement "all numbers between –3 and 4, including the 4" in inequality notation and interval notation. Then graph the inequality using both types of notation.

- The inequality notation is $-3 < x \leq 4$. The graph is shown in Figure 9-1.
- The interval notation is $(-3, 4]$. The graph is shown in Figure 9-2.

Figure 9-1: A graph of the inequality.

Figure 9-2: A graph of the interval.

Performing Operations on Inequalities

There are many similarities between working with inequalities and working with equations. The balancing part still holds. It's when operations like multiplying each side by a number or dividing each side by a number come into play that there are some differences.

The rules for operations on inequalities are given here. I'm showing the rules only for less than (<), but they also apply to greater than (>):

- If $a < b$, then $a + c < b + c$ and $a - c < b - c$. The direction of the inequality stays the same.
- If $a < b$ and c is positive, then $a \cdot c < b \cdot c$ and $\frac{a}{c} < \frac{b}{c}$. The direction of the inequality stays the same.
- If $a < b$ and c is negative, then $a \cdot c > b \cdot c$ and $\frac{a}{c} > \frac{b}{c}$. When multiplying or dividing with a negative number, the direction of the inequality symbol changes.

- If $\frac{a}{c} < \frac{b}{d}$, then $\frac{c}{a} > \frac{d}{b}$. The inequality symbol changes when you *flip* (write the reciprocals of) the fractions.

Adding and subtracting numbers to inequalities

Adding and subtracting values within inequalities works exactly the same as with equations. You keep things balanced. Let me show you how this works.

Start with an inequality statement that you can tell is true by looking at it, such as 6 is less than 10:

$$6 < 10$$

What happens if you add the same thing to each side? You can do that to an equation and not have the truth change, but what about an inequality? Add 4 to each side:

$$6 + 4 < 10 + 4$$
$$10 < 14$$

Ten is less than 14 is still a true statement. This demonstration isn't enough to prove anything, but it does illustrate a rule that is true: When you add any number to both sides of an inequality, the inequality is still correct or true. Similarly, when you subtract any number from both sides of an inequality, the inequality is still correct or true.

Multiplying and dividing inequalities

Now come the tricky operations. Multiplication and division add a new dimension to working with inequalities.

When multiplying or dividing both sides of an inequality by a positive number, the inequality remains correct or true. When multiplying or dividing both sides by a negative number, the inequality sign has to be reversed — point in the opposite direction — for the inequality to be correct or true. You can never multiply each side by 0 — that always makes it false (unless you have an *or equal to*). And, of course, you can never divide anything by 0.

Start with positive numbers, such as 20 and 12:

$$20 > 12$$

Multiply each side by 4:

$$\begin{aligned} 20 \cdot 4 &> 12 \cdot 4 \\ 80 &> 48 \end{aligned}$$

It's still true. So is there a problem?

You can see the complication with my new inequality, $10 > -3$. Multiply each side by -2:

$$\begin{aligned} 10(-2) &> -3(-2) \\ -20 &> 6 \end{aligned}$$

Oops! A negative can't be greater than a positive:

$$-20 < 6$$

Making the inequality untrue is bad news. The good news is that turning the inequality symbol around is a relatively easy way to fix this.

REMEMBER

Whenever you multiply each side of an inequality by a negative number (or divide by a negative number), turn the inequality symbol to face the opposite direction.

WARNING!

In the case of inequalities, you can neither divide nor multiply by 0. Of course, dividing by 0 is always forbidden, but you can usually multiply expressions by 0 (and get a product of 0). However, you can't multiply inequalities by 0.

Look at what happens when each side of an inequality is multiplied by 0:

$$\begin{aligned} 3 &< 7 \\ 0 \cdot 3 &< 0 \cdot 7 \\ 0 &< 0 \end{aligned}$$

No! It's just not true: Zero is not less than itself, nor is it greater than itself. So, to keep 0 from getting an inferiority or superiority complex, don't use it to multiply inequalities.

Finding Solutions for Linear Inequalities

Linear inequalities, like linear equations, are those statements in which the exponent on the variable is no more than 1. Solving linear inequalities is much like solving linear equations. The main thing to remember is to reverse the inequality symbol when you multiply or divide by a negative number — and only then. You also need to keep in mind that you don't get just a single answer to linear inequalities but a whole bunch of answers — an infinite number of answers.

EXAMPLE

Solve for the values of z in $-2(3z + 4) > 10$.

In this case, the only variable term is already on the left. A usual next step would be to distribute the –2 over the terms on the left. But, because 2 divides 10 evenly, an alternate step lets you avoid having to do the distribution.

1. **Divide each side by –2.**

   Be sure to switch the inequality symbol around.

   $$\frac{-2(3z+4)}{-2} < \frac{10}{-2}$$
   $$3z + 4 < -5$$

2. **Subtract 4 from each side.**

   $$3z + 4 - 4 < -5 - 4$$
   $$3z < -9$$

3. **Divide each side by 3.**

   $$\frac{3z}{3} < \frac{-9}{3}$$
   $$z < -3$$

Expanding to More Than Two Expressions

One big advantage that inequalities have over equations is that they can be expanded or strung out into compound statements, and you can do more than one comparison at the same time. Look at this statement:

$$2 < 4 < 7 < 11 < 12$$

You can create another true statement by pulling out any pair of numbers from the inequality, as long as you write them in the same order. They don't even have to be next to one another. For example:

$4 < 12$ $\quad$ $2 < 11$ $\quad$ $2 < 12$

One thing you can't do, though, is to mix up inequalities, going in opposite directions, in the same statement. You can't write $7 < 12 > 2$.

The operations on these compound inequality expressions use the same rules as for the linear expressions (refer to the "Performing Operations on Inequalities" section, earlier in this chapter). You just extend the process by performing the operations on each section or part.

Solve for the values of x in $-3 \le 5x + 2 < 17$.

1. **The goal is to get the variable alone in the middle. Start by subtracting 2 from each section.**

   $-3 - 2 \le 5x + 2 - 2 < 17 - 2$

   $-5 \le 5x < 15$

2. **Now divide each section by 5.**

   The number 5 is positive, so don't turn the inequality signs around.

   $\frac{-5}{5} \le \frac{5x}{5} < \frac{15}{5}$

   $-1 \le x < 3$

This says that x is greater than or equal to –1 while, at the same time, it's less than 3. Some possible solutions are: 0, 1, 2, 2.9.

3. **Check the solution using two of these possibilities.**

   If $x = 1$, then $-3 \leq 5(1) + 2 < 17$, or $-3 \leq 7 < 17$. That's true.

   If $x = 2$, then $-3 \leq 5(2) + 2 < 17$, or $-3 \leq 12 < 17$. This also works.

Taking on Quadratic and Rational Inequalities

A *quadratic inequality* is an inequality that involves a variable term with a second-degree power (and no higher powers of the variable). When solving quadratic inequalities, the rules of addition, subtraction, multiplication, and division of inequalities still hold, but the final step in the solution is different. The best way to describe how to solve a quadratic inequality is to use an example and put the rules right in the example.

EXAMPLE

Solve for x in $x^2 + 3x > 4$.

1. **Move all terms to one side.**

   First, move the 4 to the left by subtracting 4 from each side.

   $$\begin{aligned} x^2 + 3x &> 4 \\ x^2 + 3x - 4 &> 0 \end{aligned}$$

2. **Factor.**

   Factor the quadratic on the left using unFOIL.

   $(x + 4)(x - 1) > 0$

3. **Find all the values of x that make the factored side equal to 0.**

   In this case, there are two values. Using the multiplication property of zero, you get $x + 4 = 0$ or $x - 1 = 0$, which results in $x = -4$ or $x = 1$.

REMEMBER

The *multiplication property of zero* says that if the product of two or more factors equals 0, then at least one of the factors must be zero. If $xyz = 0$, then $x = 0$ or $y = 0$ or $z = 0$.

4. **Make a number line listing the values from Step 3, and determine the signs of the expression between the values on the chart.**

   When you choose a number to the left of –4, both factors are negative, and the product is positive. Between –4 and 1, the first factor is positive and the second factor is negative, resulting in a negative product. To the right of 1, both factors are positive, giving you a positive product. Just testing one of the numbers in the interval tells you what will happen to all of them.

   Figure 9-3 shows you a number line with the critical numbers in their places and the signs in the intervals between the points.

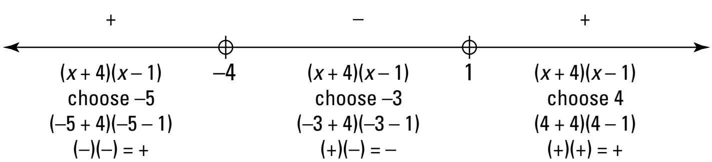

Figure 9-3: A number line helps you find the signs of the factors and their products.

5. **Determine which intervals give you solutions to the problem.**

   The values for x that work to make the quadratic $x^2 + 3x - 4 > 0$ are all the negative numbers smaller than –4 down lower to really small numbers and all the positive numbers bigger than 1 all the way up to really big numbers. The only numbers that don't work are those between –4 and 1. You write your answer as $x < -4$ or $x > 1$.

   In interval notation, the answer is $(-\infty, -4) \cup (1, \infty)$. The $\cup$ symbol is for *union,* meaning everything in either interval (one or the other) works.

Using a similar process with more than two factors

Even though this section involves problems that are *quadratic inequalities* (inequalities that have at least one squared variable term and a greater-than or less-than sign), some other types of inequalities belong in the same section because you handle them the same way as you do quadratics. You can really have any number of factors and any arrangement of factors and do the positive-and-negative business to get the answer, as I show you in the following example.

EXAMPLE

Solve for the values of x that work in $(x - 4)(x + 3)(x - 2)(x + 7) > 0$.

This problem is already factored, so you can easily determine that the numbers that make the expression equal to 0 (the critical numbers) are $x = 4$, $x = -3$, $x = 2$, $x = -7$. Put them in order from the smallest to the largest on a number line (see Figure 9-4), and test for the signs of the products in the intervals.

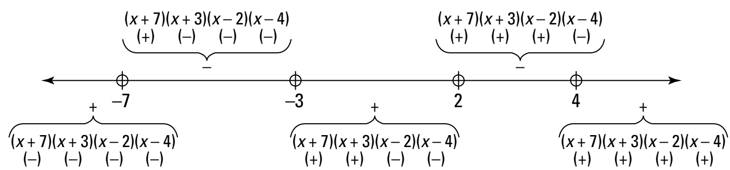

Figure 9-4: The sign changes at each critical number in this problem.

Because the original problem is looking for values that make the expression greater than 0, or positive, the solution includes numbers in the intervals that are positive. Those numbers are

- Smaller than –7
- Between –3 and 2
- Bigger than 4

The solution is written $x < -7$ or $-3 < x < 2$ or $x > 4$. In interval notation, the solution is written $(-\infty, -7) \cup (-3, 2) \cup (4, \infty)$.

Identifying the factors in fractional inequalities

Inequalities with fractions that have variables in the denominator are another special type of inequality that fits under the general heading of quadratic inequalities; they get to be in this chapter because of the way you solve them.

To solve these rational (fractional) inequalities, do somewhat the same thing as you do with the inequalities dealing with two or more factors:

1. **Find where the expression equals 0.**

   Actually, expand that to looking for, separately, what makes the numerator (top) equal to 0 and what makes the denominator (bottom) equal to 0. These are your *critical numbers.*

2. **Check the intervals between the critical numbers.**

3. **Write out the answer.**

The one big caution with rational inequalities is not to include any number in the final answer that makes the denominator of the fraction equal 0. Zero in the denominator makes it an impossible situation, not to mention an impossible fraction.

So why look at what makes the denominator 0 at all? The number 0 separates positive numbers from negative numbers. Even though the 0 itself can't be used in the solution, it indicates where the sign changes from positive to negative or negative to positive.

Solve for z in $\frac{z^2-1}{z^2-9} \le 0$.

Factor the numerator and denominator to get $\frac{(z+1)(z-1)}{(z+3)(z-3)} \le 0$. The numbers making the numerator or denominator equal to 0 are $z = +1, -1, +3, -3$. Make a number line that contains the critical numbers and the signs of the intervals (see Figure 9-5).

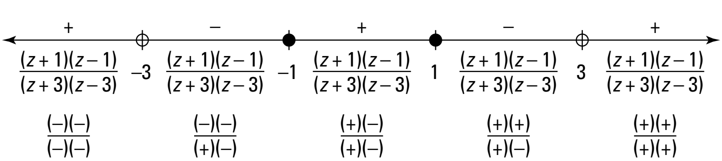

Figure 9-5: The 1 and –1 are included in the solution.

Because you're looking for values of z that make the expression negative, you want the values between –3 and –1 and those between 1 and 3. Also, you want values that make the expression equal to 0. That can only include the numbers that make the numerator equal to 0, the 1 and –1. The answer is written

$$-3 < z \leq -1 \text{ or } 1 \leq z < 3$$

In interval notation, the solution is written

$$(-3, -1] \cup [1, 3)$$

Notice that the < symbol is used by the –3 and 3 so those two numbers don't get included in the answer.